

Linear algebra reviews exercises

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Problem 1:

Find a bases for $Col(A)$ and $Nul(A)$ and then deduce the rank of A and \dim of $Nul(A)$, where

$$A = \begin{pmatrix} 1 & 2 & -4 & 4 & 6 \\ 5 & 1 & -9 & 2 & 10 \\ 4 & 6 & -9 & 12 & 15 \\ 3 & 4 & -5 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 8 & 4 & -6 \\ 0 & 2 & 3 & 4 & -1 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Problem 2:

Use the cofactor expansion to compute

$$\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix}$$

Problem 3:

Use row reduction to echelon form to compute the following determinant

$$\begin{vmatrix} 1 & a & a_2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Problem 4:

Using the Cramer's rule, determine the values of the parameter for which the system has a unique solution, and describe the solution.

$$\begin{cases} sx_1 - 2sx_2 = 1 \\ 3x_1 + 6sx_2 = 4 \end{cases}$$

Problem 5:

Let R be the triangle with vertices at (x_1, y_1) , (x_2, y_2) and (x_3, y_3) . Show that

$$\{\text{area of triangle}\} = 1/2 \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

Problem 6:

Let H and K be subspaces of a vector space V . The intersection of H and K , written $H \cap K$ is the set of v in V that belong to both H and K . Show that $H \cap K$ is a subspace of V . Give an example in \mathbb{R}^2 to show that the union of two subspaces is not, in general a subspace.

Problem 7: Let $M_{2 \times 2}$ be the vector space of all 2×2 matrices and define $T : M_{2 \times 2} \rightarrow M_{2 \times 2}$ by $T(A) = A + A^T$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1. Show that T is a linear transformation.
2. Let B be any element of $M_{2 \times 2}$ such that $B^T = B$. Find an A in $M_{2 \times 2}$ such that $T(A) = B$.
3. Show that the range of T is the set of B in $M_{2 \times 2}$ with the property that $B^T = B$.
4. Describe the kernel of T

Problem 7:

Use coordinate vectors to test the linear independence of the sets of polynomials, $\{1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3\}$. Explain your work.

Problem 8:

Explain why the space \mathbb{P} of all the polynomials is an infinite dimensional space.

Problem 9:

Verify that $\text{rank}(uv^T) \leq 1$ if $u = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ and $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Problem 10:

Let $\mathcal{D} = \{d_1, d_2, d_3\}$ and $\mathcal{F} = \{f_1, f_2, f_3\}$ be bases for a vector space V , and suppose that $f_1 = 2d_1 - d_2 + d_3$, $f_2 = 3d_2 + d_3$ and $f_3 = -3d_1 + 2d_3$.

1. Find the change-of-coordinates matrix from \mathcal{F} to \mathcal{D} .
2. Find $[x]_{\mathcal{D}}$ for $x = f_1 - 2f_2 + 2f_3$.

Problem 11:

Find the general solution of this difference equation.

$$y_k = k - 2; y_{k+2} - 4y_k = 8 - 3k$$

Problem 12:

Show that every 2×2 matrix has at least one steady-state vector. Any such matrix can be written in the form $P = \begin{pmatrix} 1 - \alpha & \beta \\ \alpha & 1 - \beta \end{pmatrix}$, where α and β are constant 0 and 1. (There are two linearly independent steady-state vectors if $\alpha = \beta = 0$. Otherwise there is only one.)

Problem 13:

Consider a matrix A with the property that the rows sum all equal the same number s . Show that s is an eigenvalue of A .

Problem 14: Orthogonally diagonalize if possible:

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

Problem 15: Let $T : \mathbb{P}_2 \rightarrow \mathbb{P}_4$ be the transformation that maps a polynomial $p(t)$ into a polynomial $p(t) + 2t^2p(t)$.

1. Find the image of $p(t) = 3 - 2t^2$.
2. Show that T is a linear transformation.
3. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

Problem 16: Classify the origin as an attractor, repeller or saddle point of the dynamical system $x_{k+1} = Ax_k$. Find the direction of the greatest attraction and/or repulsion when

$$A = \begin{pmatrix} 1.7 & 0.6 \\ -0.4 & 0.7 \end{pmatrix}$$

Problem 17: Let $W = \text{Span}\{v_1, \dots, v_p\}$. Show that if x is orthogonal to each v_j , for $1 \leq j \leq p$, then x is orthogonal to every vector in W .

Problem 18:

Show that if the vector of an orthogonal set are normalized the new set will still be orthogonal.

Problem 19:

Find the orthogonal projection of y onto $\text{Span}\{u_1, u_2\}$ where

$$y = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}, u_1 = \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Problem 20:

Find the QR factorization for A where

$$A = \begin{pmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{pmatrix}$$

Problem 21:

Find the least squares solution of $Ax = b$ where

$$A = \begin{pmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 1 \end{pmatrix}, b = \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}$$

Problem 22:

Find the least squares line $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits data points:

$$(2, 3), (3, 2), (5, 1), (6, 0)$$

Problem 23:

Let \mathbb{P}_3 have the inner product given by evaluation at $-3, -1, 1$ and 3 . Let $p_0(t) = 1$, $p_1(t) = t$ and $p_2(t) = t^2$.

1. Compute the orthogonal projection of p_2 onto the subspace spanned by p_0 and p_1 .
2. Find a polynomial q that is orthogonal to p_0 and p_1 , such that $\{p_0, p_1, q\}$ is an orthogonal basis for $\text{Span}\{p_0, p_1, p_2\}$. Scale the polynomial q so that its vector of values at $(-3, -1, 1, 3)$ is $(1, -1, -1, 1)$.

Problem 24:

Make a change of variable $x = Py$ so that transforms the quadratic form into one with no cross-product term. Write the new quadratic form and

determine if it is positive definite, negative definite or indefinite when the quadratic form is

$$Q(x) = 8x_1^2 + 6x_1x_2$$

Problem 25:

Find a SVD for

$$A = \begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix}$$